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Existence and asymptotic behavior of blow-up solutions to a class of $p(x)$-Laplacian problems ☆

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Abstract

This paper investigates the problem

$$\begin{cases} -\Delta_{p(x)}u + f(x,u) = 0 & \text{in } \Omega, \\ u(x) \to +\infty & \text{as } d(x,\partial\Omega) \to 0, \end{cases}$$

where $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. The existence of blow-up solutions is discussed, and the singularity of blow-up solutions is given.

Keywords: $p(x)$-Laplacian; Sub-solution; Super-solution; Singularity

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [13,18]). Many results have been obtained on this kind of problems, for example, [1–7,10,13–17]. In [3,6], Fan and Zhao give the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions. On the existence of solutions for $p(x)$-Laplacian problems in bounded domain, we refer to [2,4,16]. In this paper, we consider the problem

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$$\begin{cases} -\Delta_{p(x)}u + f(x,u) = 0 & \text{in } \Omega, \\ u(x) \to +\infty & \text{as } d(x, \partial\Omega) \to 0, \end{cases} \tag{P}$$

where $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, $\Omega = B(0,R) \subset \mathbb{R}^N$ is a bounded radial domain ($B(0,R) = \{x \in \mathbb{R}^N \mid |x| < R\}$). Our aim is to give the existence of blow-up solutions for problem (P), and give the singularity of blow-up solutions.

Throughout the paper, we assume that $p(x)$ and $f(x,u)$ satisfy

(H_1) $p(x) \in C^1(\overline{\Omega})$ is a radial symmetric function and satisfies

$$1 < p^- \leqslant p^+ < N, \quad \text{where } p^- = \inf_{\Omega} p(x),\ p^+ = \sup_{\Omega} p(x).$$

(H_2) $f(x,u)$ is radial with respect to x, $f(x,\cdot)$ is increasing and $f(x,0) = 0$ for any $x \in \Omega$.

(H_3) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies

$$\left|f(x,t)\right| \leqslant C_1 + C_2|t|^{\alpha(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where C_1, C_2 are positive constants, $\alpha \in C(\overline{\Omega})$ and $1 \leqslant \alpha(x) < p^*(x) := \frac{Np(x)}{N-p(x)}$.

The operator $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x) \equiv p$ (a constant), (P) is the well-known p-Laplacian problem. If $f(x,u)$ can be represented as $h(x)f(u)$, there are many papers on the blow-up solutions of p-Laplacian problems, for example, [8,9,11,12]. In the investigation of existence of blow-up solutions for the following p-Laplacian problems (p is a constant):

$$-\Delta_p u + h(x)f(u) = 0 \quad \text{in } \Omega, \tag{1}$$

the following generalized Keller–Osserman condition is crucial:

$$\int_1^\infty \frac{1}{(F(t))^{1/p}}\,dt < +\infty, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

For the problem (P), since $p(x)$ is a function, the typical form is $-\Delta_{p(x)}u + |u|^{q(x)-2}u = 0$, our results possess some difference from the generalized Keller–Osserman condition.

Because of the nonhomogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated than those of p-Laplacian ones; and another difficulty of this paper is that $f(x,u)$ can not be represented as $h(x)f(u)$.

2. Preliminary

In order to deal with $p(x)$-Laplacian problems, we need some theories on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [1,7]). Let

$$L^{p(x)}(\Omega) = \left\{u \;\middle|\; u \text{ is a measurable real-valued function, } \int_\Omega \left|u(x)\right|^{p(x)} dx < \infty\right\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{\lambda > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}.$$

The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. We call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive and uniform convex Banach space (see [1, Theorems 1.10, 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [1, Theorem 2.1]).

If $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega)$, u is called a blow-up solution of (P) if it satisfies

$$\int_Q |\nabla u|^{p(x)-2}\nabla u \nabla q \, dx + \int_Q f(x,u) q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(Q),$$

for any domain $Q \Subset \Omega$, and $\max(k-u, 0) \in W_0^{1,p(x)}(\Omega)$ for every positive integer k.

Let

$$W_{0,\mathrm{loc}}^{1,p(x)}(\Omega) = \{u \mid \text{there is an open domain } Q \Subset \Omega \text{ such that } u \in W_0^{1,p(x)}(Q)\},$$

and define

$$A : W_{\mathrm{loc}}^{1,p(x)}(\Omega) \to \big(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)\big)^*$$

as

$$\langle Au, \varphi\rangle = \int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \nabla \varphi + f(x,u)\varphi\big)\, dx,$$
$$\forall u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega), \ \forall \varphi \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega);$$

then we have

Lemma 2.1. (See [4, Theorem 3.1].) *Let $h \in W^{1,p(x)}(\Omega)$, $X = h + W_0^{1,p(x)}(\Omega)$. Then, $A : X \to (W_0^{1,p(x)}(\Omega))^*$ is strictly monotone.*

Let $g \in (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, if

$$\langle g, \varphi\rangle \geqslant 0, \quad \forall \varphi \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega), \quad \varphi \geqslant 0 \text{ a.e. in } \Omega,$$

then denote $g \geqslant 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$; correspondingly, if $-g \geqslant 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then denote $g \leqslant 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$.

Definition 2.2. Let $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega)$. If $Au \geqslant 0$ ($Au \leqslant 0$) in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then u is called a weak super-solution (weak sub-solution) of Eq. (P).

Copying the proof of [15], we have

Lemma 2.3 *(Comparison principle). Let $u, v \in W^{1,p(x)}(\Omega)$ satisfy*

$$Au - Av \geqslant 0 \quad \text{in } \left(W_0^{1,p(x)}(\Omega)\right)^*.$$

Let $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ (i.e., $u \geqslant v$ on $\partial\Omega$), then $u \geqslant v$ a.e. in Ω.

Lemma 2.4. (See [6].) *Under the conditions* (H_1) *and* (H_3), *if $u \in W^{1,p(x)}(\Omega)$ is a weak solution of $-\Delta_{p(x)}u + f(x,u) = 0$ in Ω, $u = w_0$ on $\partial\Omega$, where $w_0 \in W^{1,p(x)}(\Omega)$, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, where $\alpha \in (0,1)$ is a constant.*

3. Main results and proofs

If u is a radial solution of (P), then (P) can be transformed into

$$\begin{aligned}
&\left(r^{N-1}|u'|^{p(r)-2}u'\right)' = r^{N-1} f(r,u), \quad r \in (0,R),\\
&u(0) = u_0, \quad u'(0) = 0, \qquad u'(r) \geqslant 0 \quad \text{for } 0 < r < R.
\end{aligned} \tag{2}$$

It means that $u(r)$ is increasing.

Theorem 3.1. *If there exists $\rho \in [\frac{R}{2}, R)$ such that*

$$f(r,u) \geqslant a u^{q(r)-1} \quad (\text{as } u \to +\infty) \text{ for } r \in [\rho, R) \text{ uniformly},$$

where a is a positive constant, $q(r)$ is Lipschitz continuous and $q(r) - p(r) \geqslant \frac{1}{n}$ (where n is an integer and $n > 3$) for $r \in [\rho, R)$ uniformly, then there exists a function $\Phi_1(x)$ which satisfies $\Phi_1(x) \to +\infty$ (as $d(x, \partial\Omega) \to 0$), and such that, if u is a weak solution of problem (P) then $u(x) \leqslant \Phi_1(x)$.

Proof. Define the function $g(r,\epsilon)$ on $[0,R)$ as

$$g(r,\epsilon) = \begin{cases}
C(R-r)^{-s} + k, \quad R_0 \leqslant r < R,\\
C(R-R_0)^{-s} + k - \int_r^{R_0} [Cs(R-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}\\
\quad \times \left[\frac{(R_0)^{N-1}}{t^{N-1}} \sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}} dt, \quad \rho < r < R_0,\\
C(R-R_0)^{-s} + k - \int_\rho^{R_0} [Cs(R-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}\\
\quad \times \left[\frac{(R_0)^{N-1}}{t^{N-1}} \sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}} dt, \quad r \leqslant \rho,
\end{cases}$$

where $s = \frac{p(R)}{q(R)-p(R)}$, $R_0 \in (\rho, R)$ and $R - R_0$ is small enough, $\varepsilon = \frac{\pi}{2(R_0-\rho)}$, $0 < \epsilon < (4n^2)^{\frac{-1}{n-2}}$ is a small constant and

$$C = C_\epsilon = (1+\epsilon)\left[\frac{1}{a} s^{p(R)-1}(s+1)\big(p(R)-1\big)\right]^{\frac{1}{q(R)-p(R)}},$$

$$k = \left[C\left(\frac{R-R_0}{2}\right)^{-s}\right]^{\frac{p^+-1}{p^--1}} + \int_\rho^{R_0} \left[Cs\left(\frac{R-R_0}{2}\right)^{-s-1}\right]^{\frac{p(R_0)-1}{p(t)-1}}$$

$$\times \left[\frac{(R_0)^{N-1}}{t^{N-1}} \sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}} dt.$$

Obviously, for any positive constant ϵ, $g(r,\epsilon)\in C^1[0,R)$.

When $R_0<r<R$, we have

$$g' = Cs(R-r)^{-s-1}.$$

Then

$$\bigl(r^{N-1}|g'|^{p(r)-2}g'\bigr)' = \bigl[r^{N-1}(Cs)^{p(r)-1}(R-r)^{-(s+1)(p(r)-1)}\bigr]',$$

and

$$\begin{aligned}&\bigl(r^{N-1}|g'|^{p(r)-2}g'\bigr)'\\ &\quad = r^{N-1}(Cs)^{p(r)-1}(s+1)\bigl(p(r)-1\bigr)(R-r)^{-(s+1)(p(r)-1)-1}\bigl(1+h(r)\bigr),\end{aligned}\tag{3}$$

where

$$\begin{aligned}h(r) &= \frac{[-(s+1)p(r)]'\ln(R-r)}{(s+1)(p(r)-1)}(R-r) + \frac{[r^{N-1}(Cs)^{p(r)-1}]'}{r^{N-1}(Cs)^{p(r)-1}(s+1)(p(r)-1)}(R-r)\\ &= \frac{n[-(s+1)p(r)]'(R-r)^{\frac{1}{n}}\ln(R-r)^{\frac{1}{n}}}{(s+1)(p(r)-1)}(R-r)^{1-\frac{1}{n}}\\ &\quad + \frac{[r^{N-1}(Cs)^{p(r)-1}]'(R-r)^{\frac{1}{n}}}{r^{N-1}(Cs)^{p(r)-1}(s+1)(p(r)-1)}(R-r)^{1-\frac{1}{n}}.\end{aligned}$$

It is easy to see that there exist positive constants $A,B\geqslant 1$ (A,B depend on R,p,q,n,s) such that

$$\left|\frac{n[-(s+1)p(r)]'(R-r)^{\frac{1}{n}}\ln(R-r)^{\frac{1}{n}}}{(s+1)(p(r)-1)}\right|\leqslant A,$$
$$\left|\frac{[r^{N-1}(Cs)^{p(r)-1}]'}{r^{N-1}(Cs)^{p(r)-1}(s+1)(p(r)-1)}(R-r)^{\frac{1}{n}}\right|\leqslant B.$$

Then

$$\bigl|h(r)\bigr|\leqslant (A+B)(R-r)^{1-\frac{1}{n}}\leqslant\bigl[(A+B+1)(R-R_0)^{\frac{1}{n}}\bigr]^{n-1}.$$

If $0<R-R_0$ is small enough, we have

$$(A+B+1)(R-R_0)^{\frac{1}{n}}\leqslant\frac{\epsilon}{2}$$

and

$$(Cs)^{p(r)-1}(s+1)\bigl\{\bigl(p(r)-1\bigr)\bigl[1+h(r)\bigr]\bigr\}\leqslant aC^{q(r)-1}\left(\frac{1}{1+\epsilon}\right)^{\frac{1}{2n}}.\tag{4}$$

Since $q(r)$ is Lipschitz continuous, if $0<R-R_0$ is small enough, we have

$$(R-r)^{-(s+1)(p(r)-1)-1}\leqslant(1+\epsilon)^{\frac{1}{2n}}(R-r)^{-s(q(r)-1)},\quad \forall r\in[R_0,R).\tag{5}$$

Thus, when $0<R-R_0$ is small enough, from (3)–(5), we have

$$\begin{aligned}&\bigl(r^{N-1}|g'|^{p(r)-2}g'\bigr)'\\ &\quad\leqslant r^{N-1}(Cs)^{p(r)-1}(s+1)\bigl(p(r)-1\bigr)(R-r)^{-(s+1)(p(r)-1)-1}\bigl(1+\epsilon^{n-1}\bigr)\\ &\quad\leqslant r^{N-1}a\bigl[C(R-r)^{-s}\bigr]^{q(r)-1}\leqslant r^{N-1}ag^{q(r)-1}\leqslant r^{N-1}f(r,g),\quad\forall r\in(R_0,R).\end{aligned}$$

Then, we have

$$\big(r^{N-1}|g'|^{p(r)-2}g'\big)' \leqslant r^{N-1}f(r,g), \quad \forall r \in (R_0, R). \tag{6}$$

Obviously, if $R-R_0$ is small enough, then k is big enough, so we have

$$\begin{aligned}\big(r^{N-1}|g'|^{p(r)-2}g'\big)' &= \varepsilon\left(\frac{R_0}{r}\right)^{N-1}\big[Cs(R-R_0)^{-s}\big]^{(p(R_0)-1)}\cos\big(\varepsilon(r-\rho)\big)\\ &\leqslant r^{N-1}ak^{q(r)-1} \leqslant r^{N-1}ag^{q(r)-1} \leqslant r^{N-1}f(r,g), \quad \rho<r<R_0.\end{aligned}$$

Thus

$$\big(r^{N-1}|g'|^{p(r)-2}g'\big)' \leqslant r^{N-1}f(r,g), \quad \rho<r<R_0. \tag{7}$$

Obviously

$$\big(r^{N-1}|g'|^{p(r)-2}g'\big)' = 0 \leqslant r^{N-1}f(r,g), \quad 0\leqslant r<\rho. \tag{8}$$

Since $g(|x|,\epsilon)$ is a C^1 function on $B(0,R)$, if $0<R-R_0$ is small enough (R_0 depends on R,p,q,n,s), from (6)–(8), we can see that $g(|x|,\epsilon)$ is a super-solution of (P).

Define the function $g_m(r,\frac{\epsilon}{2})$ on $[0, R-\frac{1}{m})$ as

$$g_m\left(r,\frac{\epsilon}{2}\right)=\begin{cases} C(R-\frac{1}{m}-r)^{-s}+k, \quad R_0\leqslant r<R-\frac{1}{m},\\ k-\int_r^{R_0}[Cs(R-\frac{1}{m}-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}\big[\frac{(R_0)^{N-1}}{t^{N-1}}\sin\varepsilon(t-\rho)\big]^{\frac{1}{p(t)-1}}\,dt\\ \quad +C(R-\frac{1}{m}-R_0)^{-s}, \quad \rho<r<R_0,\\ k-\int_\rho^{R_0}[Cs(R-\frac{1}{m}-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}\big[\frac{(R_0)^{N-1}}{t^{N-1}}\sin\varepsilon(t-\rho)\big]^{\frac{1}{p(t)-1}}\,dt\\ \quad +C(R-\frac{1}{m}-R_0)^{-s}, \quad r\leqslant\rho,\end{cases}$$

where $s=\frac{p(R)}{q(R)-p(R)}$, m is a big enough integer such that $0<\frac{1}{m}\leqslant\frac{R-R_0}{2}$, $\varepsilon=\frac{\pi}{2(R_0-\rho)}$, $0<\epsilon<(4n^2)^{\frac{-1}{n-2}}$ is a small constant, $C=C_{\frac{\epsilon}{2}}$, then $g_m(|x|,\frac{\epsilon}{2})$ is a super-solution of (P) on $B(0,R-\frac{1}{m})$. If u is a solution of (P), according to the comparison principle, we get that $g_m(|x|,\frac{\epsilon}{2})\geqslant u(x)$ for any $x\in B(0,R-\frac{1}{m})$. For any $x\in B(0,R)\setminus B(0,R_0)$, we have $g_m(|x|,\frac{\epsilon}{2})\geqslant g_{m+1}(|x|,\frac{\epsilon}{2})$. Then

$$u(x)\leqslant \lim_{m\to+\infty} g_m\left(|x|,\frac{\epsilon}{2}\right), \quad \forall x\in B(0,R)\setminus B(0,R_0).$$

When $d(x,\partial\Omega)>0$ is small enough, we have

$$\lim_{m\to+\infty} g_m\left(|x|,\frac{\epsilon}{2}\right)\leqslant C_{\frac{2\epsilon}{3}}(R-r)^{-s}+k\leqslant g\big(|x|,\epsilon\big).$$

According to the comparison principle, we obtain that $g(|x|,\epsilon)\geqslant u(x)$, $\forall x\in B(0,R)$; then $\Phi_1(x)=g(|x|,\epsilon)$ is a upper control function of all of the solutions of (P).

The proof is completed. □

Theorem 3.2. *If there exists $\rho\in[\frac{R}{2},R)$ such that*

$$f(r,u)\geqslant au^{q(r)-1} \quad (\textit{as } u\to+\infty)\ \textit{for } r\in[\rho,R)\ \textit{uniformly},$$

where a is a positive constant, $q(r)$ is Lipschitz continuous and $q(r)-p(r)\geqslant\frac{1}{n}$ (where n is an integer and $n>3$) for $r\in[\rho,R)$ uniformly, then (P) possesses a blow-up solution.

Proof. Let us consider the problem

$$\begin{cases} -\Delta_{p(x)}u + f(x,u) = 0 & \text{in } \Omega, \\ u(x) = j & \text{for } x \in \partial\Omega, \end{cases} \tag{9}$$

where $j = 1, 2, \ldots$. The relative functional is

$$\varphi = \int_\Omega \frac{1}{p(x)} \left|\nabla u(x)\right|^{p(x)} dx + \int_\Omega F(x,u)\,dx,$$

where $F(x,u) = \int_0^u f(x,t)\,dt$. Since φ is coercive in $X_j := j + W_0^{1,p(x)}(\Omega)$, then φ possesses a nontrivial minimum point u_j, then problem (9) possesses a weak solution u_j. According to the comparison principle, we get $u_j(x) \leqslant u_{j+1}(x)$ for any $x \in \Omega$ and $j = 1, 2, \ldots$. Since $\Phi_1(x)$ defined in Theorem 3.1 is a super-solution, according to the comparison principle, we get $u_j(x) \leqslant \Phi_1(x)$ on Ω for all $j = 1, 2, \ldots$. Since every weak solution of (9) is a $C^{1,\alpha}_{\mathrm{loc}}$ function, and $\Phi_1(x)$ is locally bounded, similarly to the proof of Lemma 2.1 of paper [11], we can prove that $\{u_j\}$ possesses a subsequence $\{u_{j_i}\}$, such that $u_{j_i} \to u$ is a blow-up solution of (P). $\square$

Theorem 3.3. *If there exists $\rho \in [\frac{R}{2}, R)$ such that*

$$f(r,u) \leqslant bu^{q(r)-1} \quad (\textit{as } u \to +\infty) \textit{ for } r \in [\rho, R) \textit{ uniformly},$$

where b is a positive constant, $q(r)$ is Lipschitz continuous and $q(r) - p(r) \geqslant \frac{1}{n}$ (where n is an integer and $n > 3$) for $r \in [\rho, R)$ uniformly, then there exists a function $\Phi_2(x)$ which satisfies $\Phi_2(x) \to +\infty$ (as $d(x, \partial\Omega) \to 0$), and such that, if $u(x)$ is a blow-up solution of problem (P) then $u(x) \geqslant \Phi_2(x)$.

Proof. Define the function $v(r,\epsilon)$ on $B(0,R)$ as

$$v(r,\epsilon) = \begin{cases} C^*(R-r)^{-s} - k^*, \quad R_0 \leqslant r < R, \\ C^*(R-R_0)^{-s} - k^* - \int_r^{R_0} [C^* s(R-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} \left[\frac{(R_0)^{N-1}}{t^{N-1}} \sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}} dt, \\ \quad \rho < r < R_0, \\ C^*(R-R_0)^{-s} - k^* - \int_\rho^{R_0} [C^* s(R-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} \left[\frac{(R_0)^{N-1}}{t^{N-1}} \sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}} dt, \\ \quad r \leqslant \rho, \end{cases}$$

where $s = \frac{p(R)}{q(R)-p(R)}$, $R_0 \in (\rho, R)$ and $R - R_0$ is small enough, $\varepsilon = \frac{\pi}{2(R_0-\rho)}$, $0 < \epsilon < (4n^2)^{\frac{-1}{n-2}}$ is a small constant and

$$C^* = C^*_\epsilon = (1-\epsilon)\left[\frac{1}{b} s^{p(R)-1}(s+1)\big(p(R)-1\big)\right]^{\frac{1}{q(R)-p(R)}},$$

$$k^* = \left[C^*\left(\frac{R-R_0}{2}\right)^{-s}\right]^{\frac{p^+-1}{p^--1}} + \int_\rho^{R_0} \left[C^* s\left(\frac{R-R_0}{2}\right)^{-s-1}\right]^{\frac{p(R_0)-1}{p(t)-1}} \times \left[\frac{(R_0)^{N-1}}{t^{N-1}} \sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}} dt.$$

Obviously, for any positive constant $\epsilon \in (0, (4n^2)^{\frac{-1}{n-2}})$, $v(r,\epsilon) \in C^1[0,R)$.

Similar to the proof of Theorem 3.1, we have

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)' \geqslant r^{N-1}b\left[C^*(R-r)^{-s}\right]^{q(r)-1} \geqslant r^{N-1}f(r,v), \quad \forall r \in (R_0, R).$$

Since

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)' \geqslant 0 \geqslant r^{N-1}f(r,v), \quad \forall r \in [0, R_0),$$

then $v(r,\epsilon)$ is a sub-solution of (P). Define the function $v_m(r,\frac{\epsilon}{2})$ on $B(0,R)$ as

$$v_m\left(r,\frac{\epsilon}{2}\right) = \begin{cases} C^*(R+\frac{1}{m}-r)^{-s}-k^*, \quad R_0 \leqslant r < R, \\ -k^* - \int_r^{R_0}[C^*s(R+\frac{1}{m}-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}\left[\frac{(R_0)^{N-1}}{t^{N-1}}\sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}}dt \\ \quad + C^*(R+\frac{1}{m}-R_0)^{-s}, \quad \rho < r < R_0, \\ -k^* - \int_\rho^{R_0}[C^*s(R+\frac{1}{m}-R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}\left[\frac{(R_0)^{N-1}}{t^{N-1}}\sin\varepsilon(t-\rho)\right]^{\frac{1}{p(t)-1}}dt \\ \quad + C^*(R+\frac{1}{m}-R_0)^{-s}, \quad r \leqslant \rho, \end{cases}$$

where $C^* = C^*_{\frac{\epsilon}{2}}$, $\varepsilon = \frac{\pi}{2(R_0-\rho)}$. We can see that $v_m(r,\frac{\epsilon}{2})$ is a sub-solution of (P) on $B(0,R)$, according to the comparison principle, we get that $v_m(r,\frac{\epsilon}{2}) \leqslant u(x)$ for any $x \in B(0,R)$. For any $x \in B(0,R)\setminus B(0,R_0)$, we have $v_m(|x|,\frac{\epsilon}{2}) \leqslant v_{m+1}(|x|,\frac{\epsilon}{2})$. Then

$$u(x) \geqslant \lim_{m\to+\infty} v_m\left(|x|,\frac{\epsilon}{2}\right), \quad \forall x \in B(0,R)\setminus B(0,R_0).$$

When $d(x,\partial\Omega)$ is small enough, we have

$$\lim_{m\to+\infty} v_m\left(|x|,\frac{\epsilon}{2}\right) \geqslant C^*_{\frac{2\epsilon}{3}}(R-r)^{-s} - k^* \geqslant v\left(|x|,\epsilon\right).$$

According to the comparison principle, we obtain that $v(|x|,\epsilon) \leqslant u(x)$, $\forall x \in B(0,R)$; then $\Phi_2(x) = v(|x|,\epsilon)$ is the lower control function of all of the solutions of (P). □

Definition 3.1. If u is a blow-up solution of (P) that satisfies

$$\lim_{d(x,\partial\Omega)\to 0} \frac{u(x)}{\beta(d(x,\partial\Omega))^{-s}} = 1,$$

where β and s are positive constants, then we say that the singularity of u is $\beta(d(x,\partial\Omega))^{-s}$.

Theorem 3.4. *If*

$$\lim_{r\to R^-}\lim_{u\to+\infty}\frac{f(r,u)}{u^{q(r)-1}} = a,$$

where a is a positive constant, $q(r)$ is Lipschitz continuous and $q(r)-p(r) \geqslant \frac{1}{n}$ (where n is an integer and $n > 3$), then the singularity of the blow-up solution u of (P) is $\beta(d(x,\partial\Omega))^{-s}$, where

$$s = \frac{p(R)}{q(R)-p(R)}, \quad \beta = C_0 = \left(\frac{1}{a}s^{p(R)-1}(s+1)\left(p(R)-1\right)\right)^{\frac{1}{q(R)-p(R)}}.$$

Proof. It can be obtained easily from Theorems 3.2 and 3.3. □

Theorem 3.5. *If there exists $\rho \in [\frac{R}{2}, R)$ such that*

$$f(r,u) \leqslant bu^{q-1} \quad (\text{as } u \to +\infty) \text{ for } r \in [\rho, R) \text{ uniformly,} \tag{10}$$

where b, q are positive constants, and $1 < q < p(R)$, then (P) does not have a radial blow-up solution.

Proof. Suppose, for contradiction, that (P) has a radial blow-up solution. According to the continuity of $p(x)$ and (10), there exists a positive $\rho_1 \in [\rho, R)$ such that

$$\theta := \sup_{r\in[\rho_1,R)} \frac{q-1}{p(r)-1} < 1, \tag{11}$$

$$f\big(r,u(r)\big) \leqslant bu^{q-1}(r), \quad \forall r \in [\rho_1, R), \tag{12}$$

$$1 \leqslant u(r), \quad \forall r \in [\rho_1, R),$$

and

$$1 \leqslant \frac{b}{N}\rho_1^N u^{q-1}(\rho_1). \tag{13}$$

Integrating (2) from 0 to r, we have

$$r^{N-1}|u'|^{p(r)-2}u' = \int_0^r s^{N-1} f\big(s,u(s)\big)\,ds. \tag{14}$$

Since u is increasing and $f(r,u)$ is increasing with respect to u, we have

$$f\big(r,u(r)\big) \leqslant f\big(r,u(\rho_1)\big), \quad \forall r \in [r, \rho_1]. \tag{15}$$

Combining the continuity of f and (15), we get

$$\int_0^{\rho_1} s^{N-1} f\big(s,u(s)\big)\,ds \leqslant \rho_1^{N-1} \int_0^{\rho_1} f\big(s,u(\rho_1)\big)\,ds := C_4. \tag{16}$$

Since $u(r)$ is increasing, from (12)–(14) and (16), we have

$$\begin{aligned}
r^{N-1}|u'|^{p(r)-2}u' &= \int_0^{\rho_1} s^{N-1} f\big(s,u(s)\big)\,ds + \int_{\rho_1}^{r} s^{N-1} f\big(s,u(s)\big)\,ds \\
&\leqslant C_4 + \int_{\rho_1}^{r} s^{N-1} bu^{q-1}(s)\,ds \\
&\leqslant C_4 + bu^{q-1}(r)\int_0^r s^{N-1}\,ds \\
&\leqslant C_4 \frac{b}{N}\rho_1^N u^{q-1}(\rho_1) + \frac{b}{N} r^N u^{q-1}(r) \\
&\leqslant (C_4+1)\frac{b}{N} r^N u^{q-1}(r), \quad \forall r \in [\rho_1, R).
\end{aligned}$$

Then

$$r^{N-1}|u'|^{p(r)-2}u' \leqslant (C_4+1)\frac{b}{N}r^N u^{q-1}(r), \quad \forall r \in [\rho_1, R). \tag{17}$$

From (11) and (17), for any $r \in [\rho_1, R)$ we have

$$u' \leqslant \left[\left(C_4+1\right)\frac{b}{N}ru^{q-1}(r)\right]^{\frac{1}{p(r)-1}} \leqslant \left[(C_4+1)\frac{b}{N}r\right]^{\frac{1}{p(r)-1}} u^{\theta}(r).$$

Then

$$\frac{u'}{u^{\theta}(r)} \leqslant \left[(C_4+1)\frac{b}{N}r\right]^{\frac{1}{p(r)-1}}, \quad \forall r \in [\rho_1, R). \tag{18}$$

Integrating (18) from ρ_1 to r, yields

$$\int_{\rho_1}^{r} \frac{u'}{u^{\theta}(s)}\,ds \leqslant \int_{\rho_1}^{R} \left[(C_4+1)\frac{b}{N}r\right]^{\frac{1}{p(r)-1}} dr. \tag{19}$$

The left side of (19) tends to $+\infty$ as $r \to R^-$, it is a contradiction. The proof is completed. □

Note. In (1), assuming that $h(x)f(u) = |u|^{q-2}u$, yields

(i) if $q > p$ then (1) has blow-up solution;
(ii) if $q < p$ then (1) does not have radial blow-up solution.

For problem (P), assuming that $f(x,u) = |u|^{q(x)-2}u$, where $q(x)$ is Lipschitz continuous and radial, yields

(i) if $q(R) > p(R)$ then (P) has blow-up solution;
(ii) if $q(R) < p(R)$ then (P) does not have radial blow-up solution.

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